PRINCIPLES OF ANALYSIS
TOPIC IV: SEQUENCES

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Abstract. We define sequences, convergence of sequences, bounded sequences, monotone sequences, limits superior and inferior, and Cauchy sequences. We prove the Monotone Convergence Principle and the Cauchy Convergence Criterion.

1. Sequences

Definition 1. Let A be a set. A *sequence* in A is a function $a : \mathbb{N} \to A$. We write a_n to mean $a(n)$, and we write $(a_n)_{n=1}^{\infty}$, or simply (a_n), to denote the function a.

We are primarily interested in sequences of real numbers, i.e., sequences in $\mathbb{R}$.

Definition 2. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and let $p \in \mathbb{R}$. We say that $(a_n)_{n=1}^{\infty}$ *converges to* p

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \ni \; n \geq N \; \Rightarrow \; |a_n - p| < \epsilon.$$

In this case, we say that p is a *limit point* of $(a_n)_{n=1}^{\infty}$.

Proposition 1. *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ and let $p_1, p_2 \in \mathbb{R}$. If $(a_n)_{n=1}^{\infty}$ converges to p_1 and to p_2, then $p_1 = p_2$.*

Proof. Suppose not, and set $d = |p_1 - p_2|$; then d is positive. Let $\epsilon = \frac{d}{4}$. Then by definition of limit, there exist positive integers N_1 and N_2 such that $n \geq N_1$ implies that $|a_n - p_1| < \epsilon$, and $n \geq N_2$ implies that $|a_n - p_2| < \epsilon$.

Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$,

$$\begin{aligned} d &= |p_1 - p_2| \\ &= |p_1 - a_n + a_n - p_2| \\ &= |p_1 - a_n| + |a_n - p_2| \qquad \text{by the Triangle Inequality} \\ &= |a_n - p_1| + |a_n - p_2| \\ &\leq \epsilon + \epsilon \\ &= \frac{d}{2}. \end{aligned}$$

This is a contradiction; thus $p_1 = p_2$. □

Thus limits are unique when they exist, justifying the article *the* limit instead of "a limit point". We write $p = \lim_{n\to\infty} a_n$, or simply $p = \lim a_n$, or even $a_n \to p$ to denote the fact that $(a_n)_{n=1}^{\infty}$ converges to p. If a sequence has a limit, we say that it is *convergent*; otherwise it is *divergent*.

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Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. The *image* of $(a_n)_{n=1}^{\infty}$ is the image of the sequence as a function, that is, it is the set

$$\{a_n \mid n \in \mathbb{N}\}.$$

Note that there is much more information in a sequence than in its image; for example, the sequences $(1+(-1)^n)_{n=1}^{\infty}$ and $(0, 2, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 2, \dots)$ have the same image; the common image is $\{0, 2\}$, a set containing two elements.

2. Arithmetic of Sequences

Proposition 2. *Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$, and let $k \in \mathbb{R}$. Then the sequence $(ka_n)_{n=1}^{\infty}$ converges, and*

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n.$$

Proof. Let $\epsilon > 0$, and let $p = \lim_{n\to\infty} a_n$. Since $a_n \to p$, there exists $N \in \mathbb{N}$ such that

$$|a_n - p| < \frac{\epsilon}{k}.$$

Then

$$|ka_n - kp| < \epsilon.$$

$\square$

Proposition 3. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences of real numbers. Then the sequence $(a_n + b_n)_{n=1}^{\infty}$ converges, and*

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Proposition 4. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences of real numbers. Then the sequence $(a_n b_n)_{n=1}^{\infty}$ converges, and*

$$\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

Proposition 5. *Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence of nonzero real numbers whose limit is not zero. Then the sequence $(\frac{1}{a_n})_{n=1}^{\infty}$ converges, and*

$$\frac{1}{\lim_{n\to\infty} a_n} = \lim_{n\to\infty} \left(\frac{1}{a_n}\right).$$

3. Bounded Sequences

Definition 3. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. We say that (a_n) is *bounded above* if there exists $a \in \mathbb{R}$ such that $a \geq s_n$ for every $n \in \mathbb{N}$. We say that (a_n) is *bounded below* if there exists $b \in \mathbb{R}$ such that $b \leq a_n$ for every $n \in \mathbb{N}$. We say that $(a_n)_{n=1}^{\infty}$ is *bounded* if it is bounded above and bounded below.

Equivalently, $(a_n)_{n=1}^{\infty}$ is bounded if there exists $b > 0$ such that $a_n \in [-b, b]$ for every $n \in \mathbb{N}$.

Proposition 6. *Every convergent sequence in $\mathbb{R}$ is bounded.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence with limit p. Let N be so large that for $n \geq N$ we have $|a_n - p| < 1$. And $|p|$ to both sides of this inequality and apply the triangle inequality to get, for every $n \geq N$,

$$|a_n| \leq |a_n - p| + |p| < 1 + |p|.$$

There are only finitely many terms of the sequence between a_1 and a_{N-1}; set

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |p|\}.$$

Then $M \geq a_n$ for every $n \in \mathbb{N}$, so $(a_n)_{n=1}^{\infty}$ is bounded. □

Proposition 7. *Let $(s_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ which converges to p, and let $a, b \in \mathbb{R}$ with $a < b$.*

(a) *If $s_n \geq a$ for every $n \in \mathbb{N}$, then $p \geq a$.*
(b) *If $s_n \leq b$ for every $n \in \mathbb{N}$, then $p \leq b$.*
(c) *If $s_n \in [a, b]$ for every $n \in \mathbb{N}$, then $p \in [a, b]$.*

Proof. In this proof, we use the fact that if $x \leq y + \epsilon$ for every $\epsilon > 0$, then $x \leq y$. To see this, suppose that $x > y$, and let $\epsilon = \frac{x-y}{2}$; then $y + \epsilon = x - \epsilon$, so $x > y + \epsilon$.

Suppose that $s_n \geq a$ for every $n \in \mathbb{N}$. To show that $a \leq p$, it suffices to show that $a \leq p + \epsilon$ for every $\epsilon > 0$. Thus let $\epsilon > 0$; since (s_n) converges to p, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |s_n - p| < \epsilon$. Thus $-\epsilon < s_n - p < \epsilon$, so $s_n < p + \epsilon$. Since $a \leq s_n$, transitivity of order implies that $a < p + \epsilon$. Since this is true for every $\epsilon > 0$, we have $a \leq p$.

That $p \leq b$ can be proved similarly.

Finally, if $s_n \in [a, b]$, we have $a \leq s_n \leq b$ for every $n \in \mathbb{N}$. Combining parts **(a)** and **(b)** tells us that $a \leq p \leq b$, which is equivalent to $p \in [a, b]$. □

Proposition 8. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences in $\mathbb{R}$ such that $a_n \leq b_n$ for every $n \in \mathbb{N}$. If they both converge, then $\lim a_n \leq \lim b_n$.*

Proof. Let $a = \lim a_n$ and $b = \lim b_n$; suppose by way of contradiction that $b < a$. Set $\epsilon = \frac{b-a}{2}$; then there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - a| < \epsilon/2$, and there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|b_n - b| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$; then by an application of the triangle inequality, $b_n < a_n$, a contradiction. □

Proposition 9. (Squeeze Law)
Let (a_n), (b_n), and (s_n) be sequences in $\mathbb{R}$ such that $a_n \leq s_n \leq b_n$ for all $n \in \mathbb{N}$. If $\lim a_n = \lim b_n = p$, then (s_n) converges to p.

Proof. Let $\epsilon > 0$. Note that for any $n \in \mathbb{N}$, since $a_n \leq s_n \leq b_n$ we have

$$|s_n - a_n| = s_n - a_n \leq b_n - a_n = |b_n - a_n|.$$

Since $\lim a_n = p$, there exists $N_1 \in \mathbb{N}$ such that $|a_n - p| < \frac{\epsilon}{3}$ for $n \geq N_1$.
Since $\lim b_n = s$, there exists $N_2 \in \mathbb{N}$ such that $|b_n - p| < \frac{\epsilon}{3}$ for $n \geq N_2$.
Let $N = \max\{N_1, N_2\}$. Now for $n \geq N$, we have

$$|b_n - a_n| = |b_n - p + p - a_n| \leq |b_n - p| + |a_n - p| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

Then for $n \geq N$, we have

$$|s_n - p| = |s_n - a_n + a_n - p| \leq |s_n - a_n| + |a_n - p| \leq |b_n - a_n| + |a_n - p| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This shows that $\lim s_n = p$. □

4. Monotone Sequences

Definition 4. Let $(s_n)_{n=1}^{\infty}$ be a sequence of real numbers.
We say that (s_n) is *increasing* if

$$m \leq n \Rightarrow s_m \leq s_n.$$

We say that (s_n) is *decreasing* if

$$m \leq n \Rightarrow a_m \geq a_n.$$

We say that (s_n) is *monotone* if it is either increasing or decreasing.

Note that to check if the sequence (s_n) is increasing, if suffices to check that $s_{n+1} \geq s_n$ for every $n \in \mathbb{N}$. In this case, the definition above will follow by induction. The analogous comment holds for the condition of decreasing.

Theorem 1. (Monotone Convergence Principle)
Every bounded monotone sequence of real numbers converges.

Proof. Suppose that $(s_n)_{n=1}^{\infty}$ is bounded. Also assume that it is increasing; the proof for decreasing will be analogous. Let $S = \{s_n \mid n \in \mathbb{N}\}$ be the image of the sequence, and set $u = \sup S$. Since S is bounded, $u \in \mathbb{R}$. Clearly $s_n \leq u$ for every $n \in \mathbb{N}$. We show that $\lim s_n = u$.

Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for S, there exists $s \in S$ such that $u - \epsilon < s \leq u$. Now $s = s_N$ for some $N \in \mathbb{N}$, and since $(s_n)_{n=1}^{\infty}$ is increasing, we have $u - \epsilon < s_n < u$ for every $n \geq N$. Thus $|s_n - u| < \epsilon$ for $n \geq N$; this shows that (s_n) converges to u. □

5. Limits Superior and Inferior

Proposition 10. *Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Set*

$$u_N = \sup\{s_n \mid n \geq N\} \quad \textit{and} \quad v_N = \inf\{s_n \mid n \geq N\}.$$

Then $(u_n)_{n=1}^{\infty}$ is a bounded decreasing sequence and $(v_n)_{n=1}^{\infty}$ is a bounded increasing sequence. Each of these sequences converges.

Proof. Since (s_n) is a bounded sequence, the sets $\{s_n \mid n \geq N\}$ are bounded sets, so u_N and v_N exist as real numbers for all $N \in \mathbb{N}$, and in fact if $S = \{s_n \mid n \in \mathbb{N}\}$, then $\inf S \leq v_N \leq u_N \leq \sup S$ for every $N \in \mathbb{N}$. Thus the sequences (u_N) and (v_N) are bounded sequences.

To show that these sequences are monotone, we use the general fact that if $A, B \subset \mathbb{R}$ and $B \subset A$, then $\sup B \leq \sup A$ and $\inf B \geq \inf A$.

In our case, select $N \in \mathbb{N}$ and let $A = \{s_n \mid n \geq N\}$ and $B = \{s_n \mid n \geq N+1\}$. Then $B \subset A$, so $\sup B \leq \sup A$, which is to say, $u_{N+1} \leq u_N$. Thus (u_N) is a decreasing sequence. Similarly, (v_N) is an increasing sequence.

Thus (u_N) and (v_N) are bounded monotone sequences, and so are convergent by the Monotone Convergence Principal. $\square$

Definition 5. Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Define the *limit superior* of (s_n) to be

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n \mid n \geq N\}$$

and the *limit inferior* of (s_n) to be

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n \mid n \geq N\}.$$

Proposition 11. *Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers.*
Then $\liminf s_n \leq \limsup s_n$.

Proof. For every $N \in \mathbb{N}$, we have $\inf\{s_n \mid n \geq N\} \leq \sup\{s_n \mid n \geq N\}$. The result follows from Proposition 8. $\square$

Proposition 12. *Let $(s_n)_{n=1}^\infty$ be a sequence of real numbers.*

(a) *If (s_n) converges to s, then $\liminf s_n = s = \limsup s_n$.*
(b) *If $\liminf s_n = \limsup s_n$, then (s_n) converges.*

Proof. We again use the fact that if $x \leq y + \epsilon$ for every $\epsilon > 0$, then $x \leq y$.

Suppose that $(s_n)_{n=1}^\infty$ converges to a real number s. Let $\epsilon > 0$. We wish to show that $\limsup s_n \leq s + \epsilon$ for every $\epsilon > 0$, whence $\limsup s_n \leq s$.

Since $s_n \to s$, there exists $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ for $n \geq N$. It follows that $\sup\{s_n \mid n \geq N\} < s + \epsilon$. Since $(\sup\{s_n \mid n \geq N\})_{N=1}^\infty$ is a decreasing sequence, we have $\limsup s_n < s + \epsilon$. Therefore $\limsup s_n \leq s$.

Similarly, $s \leq \liminf s_n$, so

$$s \leq \liminf s_n \leq \limsup s_n \leq s,$$

so

$$\liminf s_n = s = \limsup s_n.$$

Now suppose that $\liminf s_n = \limsup s_n$, and label this common value s. We want to show that $\lim s_n = s$.

Let $\epsilon > 0$. Since $s = \limsup s_n$, there exists $N_1 \in \mathbb{N}$ such that

$$|\sup\{s_n \mid n \geq N_1\} - s| < \epsilon.$$

In particular, $\sup\{s_n \mid n \geq N_1\} < s + \epsilon$, so $s_n < s + \epsilon$ for $n \geq N_1$. Similarly, since $s = \liminf s_n$, there exists $N_2 \in \mathbb{N}$ such that $s_n > s - \epsilon$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we have $s - \epsilon < s_n < s + \epsilon$, that is, $|s_n - s| < \epsilon$. Thus $s_n \to s$. $\square$

6. Cauchy Sequences

Let $(s_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that $(s_n)_{n=1}^{\infty}$ is a *Cauchy sequence* if

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \ni \ m, n \geq N \Rightarrow |s_m - s_n| < \epsilon.$$

Proposition 13. *Let $(s_n)_{n=1}^{\infty}$ be a Cauchy sequence. Then $(s_n)_{n=1}^{\infty}$ is bounded.*

Proof. Since $(s_n)_{n=1}^{\infty}$ is Cauchy, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|s_m - s_n| < 1$. In particular, for every $n \geq N$, we have $|s_n - s_N| < 1$. Set

$$M = \max\{s_1, s_2, \dots, s_{N-1}, s_N + 1\}.$$

Then $s_n \in [-M, M]$ for every $n \in \mathbb{N}$. □

Theorem 2. (Cauchy Convergence Criterion)
A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proof. We prove each direction of the double implication.

($\Rightarrow$) Assume that the sequence (s_n) is convergent. Let $\epsilon > 0$, and set $s = \lim s_n$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|s_n - s| < \epsilon/2$. Then for $m, n \geq N$, we have

$$\begin{aligned}
|s_m - s_n| &= |s_m - s + s - s_n| \\
&= |s_m - s| + |s_n - s| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

($\Leftarrow$) Assume that the sequence (s_n) is a Cauchy sequence. Then it is bounded, and so its limit superior and inferior exist as real numbers. By a previous proposition, it suffices to show that $\liminf s_n = limsups_n$.

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|s_m - s_n| < \epsilon$. In particular, $|s_n - s_N| < \frac{\epsilon}{2}$ for all $n \geq N$, so $s_N + \frac{\epsilon}{2}$ is an upper bound for $\{s_n \mid n \geq N\}$. Thus $\sup\{s_n \mid n \geq N\} \leq s_N + \frac{\epsilon}{2}$, and therefore $\limsup s_n \leq s_N + \frac{\epsilon}{2}$. Similarly $\liminf s_n \geq s_N - \frac{\epsilon}{2}$. Rearranging these inequalities gives

$$\limsup s_n - \frac{\epsilon}{2} \leq s_N \leq \liminf s_n + \frac{\epsilon}{2},$$

or

$$0 \leq \limsup s_n - \liminf s_m < \epsilon.$$

Since ϵ is arbitrary, we have $\limsup s_n = \liminf s_n$. □

7. Problems

Problem 1. Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence of real numbers, and let $A = \{a_n \mid n \in \mathbb{N}\}$. Show that $\lim_{n\to\infty} a_n \leq \sup A$.

Problem 2. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. Show that if (a_n) converges to p, then $p \in [a, b]$.

Problem 3. Let $(s_n)_{n=1}^{\infty}$ be a sequence of nonzero real numbers such that $\lim_{n\to\infty} |s_n|$ converges to a positive real number. Show that there exists $m > 0$ such that $|s_n| > m$ for all n. (This is a Lemma for Proposition 5).

Problem 4. Let (s_n) and (t_n) be sequences in $\mathbb{R}$ such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Show that $\lim s_n = 0$.

Solution. Since $|s_n| \leq t_n$, we have $-t_n \leq s_n \leq t_n$.

Let $\epsilon > 0$ and let N be so large that $|t_n - 0| < \epsilon$ for $n > N$. Since

$$|t_n - 0| = |t_n| = |-t_n| = |-t_n - 0|,$$

then $|-t_n - 0| < \epsilon$ for $n > N$. Thus $\lim -t_n = 0$.

The result follows by the Squeeze Law. $\square$

Problem 5. Let (a_n) and (b_n) be sequences in $\mathbb{R}$ such that (a_n) is bounded and $\lim b_n = 0$. Show that $\lim a_n b_n = 0$.

Solution. Let $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Since $\lim b_n = 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|b_n - 0| < \frac{\epsilon}{M}$. Then for $n > N$, we have

$$|a_n b_n - 0| = |a_n||b_n| \leq M\frac{\epsilon}{M} = \epsilon.$$

Thus $\lim a_n b_n = 0$. $\square$

Problem 6. Construct sequences (a_n) and (b_n) of positive real numbers, with $c_n = a_n b_n$, satisfying

- **(0)** $\lim_{n\to\infty} b_n = 0$;
- **(1)** $\liminf c_n = 1$;
- **(2)** $\limsup c_n = 2$.

Problem 7. Let (a_n) be a sequence of positive real numbers satisfying $a_{n+1}^2 = a_n$. Show that (a_n) converges to 1.

Definition 6. Let $A \subset \mathbb{R}$ be an open interval. A function $f : A \to \mathbb{R}$ is called a *contraction* if there exists $M \in \mathbb{R}$ such that $|f(a) - f(b)| \leq M|a - b|$ for any $a, b \in U$.

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction. Let (a_n) be a sequence of real numbers which converges to $p \in \mathbb{R}$. Show that $\lim f(a_n) = f(L)$.

Solution. Let $\epsilon > 0$. Since f is a contraction, there exists $M \in \mathbb{R}$ such that $|f(a) - f(b)| < M|a - b|$ for all $a, b \in \mathbb{R}$.

Since (a_n) converges to p, there exists $N \in \mathbb{N}$ such that $|a_n - p| < \frac{\epsilon}{M}$ for all $n > N$. Since f is a contraction,

$$|f(a_n) - f(p)| < M|a_n - p| < M\frac{\epsilon}{M} = \epsilon$$

for all $n > N$. Thus $f(a_n) \to f(p)$. $\square$

Problem 9. Let (s_n) be a sequence in $\mathbb{R}$.
Show that $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

Solution. Suppose that $\limsup |s_n| = 0$. Since $(|s_n|)$ is a sequence of nonnegative numbers, so is every subsequence. The limit of a convergent sequence of nonnegative numbers is nonnegative, and since $\liminf |s_n|$ is the limit of a subsequence, we have

$$0 \leq \liminf |s_n| \leq \limsup |s_n| = 0.$$

Thus $\liminf |s_n| = \limsup |s_n| = 0$, so $(|s_n|)$ converges to zero. Therefore (s_n) converges to zero.

Suppose that $\lim s_n = 0$. Then $\lim |s_n| = 0$; this tacitly implies that $(|s_n|)$ converges, so its limit superior is equal to its limit. That is, $\limsup |s_n| = 0$. $\square$

Problem 10. Let (s_n) and (t_n) be sequences in $\mathbb{R}$.
Show that $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$.

Lemma 1. *Let $A, B \subset \mathbb{R}$ be bounded above, with $B \subset A$. Then $\sup B \leq \sup A$.*

Lemma 2. *Let $A, B \subset \mathbb{R}$ be bounded above, and set*

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Then $\sup(A + B) = \sup A + \sup B$.

Solution. Let $S_m = \{s_n \mid n > m\}$, $T_m = \{t_n \mid n > m\}$, and $U_m = \{s_n + t_n \mid n > m\}$. We have $\sup(S_m + T_m) = \sup S_m + \sup T_m$ by Lemma 2. But $U_m \subset S_m + T_m$, so $\sup U_m \leq \sup S_m + \sup T_m$ by Lemma 1. Thus

$$\begin{aligned}
\limsup(s_n + t_n) &= \lim(\sup U_m) \\
&\leq \lim(\sup S_m + \sup T_m) \\
&= \lim(\sup S_m) + \lim(\sup T_m) \\
&= \limsup s_n + \limsup t_n.
\end{aligned}$$

$\square$

Problem 11. Let (s_n) and (t_n) be bounded sequences over nonnegative real numbers.
Show that $\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n)$.

Lemma 3. *Let $S, T \subset \mathbb{R}$ be bounded sets of nonnegative real numbers. Define*
$$ST = \{st \mid s \in S, t \in T\}.$$
Then $\sup ST = (\sup S)(\sup T)$.

Proof of Lemma. Let $st \in ST$. Then $s \leq \sup S$ and $t \leq \sup T$. Since s and t are nonnegative, $st \leq (\sup S)(\sup T)$. Thus $\sup ST \leq (\sup S)(\sup T)$.

Suppose $\sup ST < (\sup S)(\sup T)$. Then $\sup ST/\sup T < \sup S$. Select $s \in S$ such that $\sup ST/\sup T < s < \sup S$. Then $\sup ST/s < \sup T$. Select $t \in T$ such that $\sup ST/s < t < \sup T$. Then $\sup ST < st$, a contradiction. □

Solution. Let $S_m = \{s_n \mid n > m\}$, $T_m = \{t_n \mid n > m\}$, and $U_m = \{s_n t_n \mid n > m\}$. We have $\sup(S_m T_m) = (\sup S_m)(\sup T_m)$ by Lemma 3. But $U_m \subset S_m T_m$, so $\sup U_m \leq \sup S_m \sup T_m$ by Lemma 1. Thus
$$\begin{aligned}\limsup(s_n t_n) &= \lim(\sup U_m) \\ &\leq \lim(\sup S_m \sup T_m) \\ &= \lim(\sup S_m)\lim(\sup T_m) \\ &= \limsup(s_n)\limsup(t_n).\end{aligned}$$
□

Problem 12. Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let $v = \liminf s_n$ and $u = \limsup s_n$. Show that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $s_n \in (v - \epsilon, u + \epsilon)$.

Problem 13. Let (s_n) be a sequence of real numbers which converges to $s \in \mathbb{R}$. Let $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i$. Show that (σ_n) converges to s.

Solution. Let $\tau_n = \sigma_n - s$. It suffices to show that (τ_n) converges to zero. Note that
$$\tau_n = \frac{1}{n}\sum_{i=1}^{n} s_i - \frac{ns}{n} = \frac{1}{n}\sum_{i=1}^{n}(s_i - s).$$

Let $N_0 \in \mathbb{N}$ be so large that $|s_n - s| < \frac{\epsilon}{2}$ for all $n > N_0$. Let $M = \sum_{i=1}^{N} |s_i - s|$. Then for $n > N_0$, we have
$$\begin{aligned}|\tau_n| &\leq \frac{M}{n} + \frac{1}{n}\sum_{i=N_0+1}^{n} |s_n - s| && \text{by } \Delta\text{-inequality} \\ &< \frac{M}{n} + \frac{1}{n}(n - N_0)\frac{\epsilon}{2} && \text{summing } n - N_0 \text{ small numbers} \\ &< \frac{M}{n} + \frac{\epsilon}{2} && \text{since } \frac{n - N_0}{n} \leq 1.\end{aligned}$$
Now select $N \in \mathbb{N}$ with $N > N_0$ which is so large that $\frac{M}{n} < \frac{\epsilon}{2}$. Then for $n > N$, we have $|\tau_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This shows that $|\tau_n| \to 0$ as $n \to \infty$. Thus $\lim \tau_n = 0$. □

Problem 14. Let (a_n) and (b_n) be a sequences of real numbers we converge to a and b respectively. Let

$$\mu_n = \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_{n-1} b_2 + a_n b_1}{n}.$$

Show that (μ_n) converges to ab.

Solution. Let $\nu_n = \mu_n - ab$. It suffices to show that (ν_n) converges to zero.

Since (a_i) is a convergent sequence, is bounded; select $M > 0$ such that $|a_i| \leq M$. Also note that for any sequence (s_i), we have $\sum_{i=1}^n s_{n-i+1} = \sum_{i=1}^n s_i$; this follows from inductive use of commutativity.

Now

$$\begin{aligned}
|\nu_n| &= \frac{1}{n} |\sum_{i=1}^n a_i b_{n-i+1} - \frac{nab}{n}| \\
&= \frac{1}{n} |\sum_{i=1}^n (a_i b_{n-i+1} - ab)| \\
&\leq \frac{1}{n} \sum_{i=1}^n |a_i b_{n-i+1} - ab| \\
&= \frac{1}{n} \sum_{i=1}^n |a_i b_{n-i+1} - a_i b + a_i b - ab| \\
&\leq \frac{\sum_{i=1}^n |a_i b_{n-i+1} - a_i b|}{n} + \frac{\sum_{i=1}^n |a_i b - ab|}{n} \\
&\leq M \frac{\sum_{i=1}^n |b_{n-i+1} - b|}{n} + b \frac{\sum_{i=1}^n |a_i - a|}{n} \\
&= M \frac{\sum_{i=1}^n |b_i - b|}{n} + b \frac{\sum_{i=1}^n |a_i - a|}{n}.
\end{aligned}$$

Let $\tau_n = M \frac{\sum_{i=1}^n |b_i - b|}{n} + b \frac{\sum_{i=1}^n |a_i - a|}{n}$. By the Problem 13,

$$\lim_{n\to\infty} \tau_n = M \lim \frac{\sum_{i=1}^n |b_i - b|}{n} + b \lim \frac{\sum_{i=1}^n |a_i - a|}{n} = M \cdot 0 + b \cdot 0 = 0.$$

Since $0 \leq |\nu_n| \leq \tau_n$ and $\lim \tau_n = 0$, we have $|\nu_n| \to 0$ so $\lim \nu_n = 0$. □

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